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1981 J. Phys. A: Math. Gen. 14 1029

(http://iopscience.iop.org/0305-4470/14/5/022)

J. Phys. A: Math. Gen. **14** (1981) 1029–1037. Printed in Great Britain

A Fourier series approach to Burgers' equation

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Received 13 August 1980, in final form 4 November 1980

Abstract. Burgers' equation is known to have wide application and attempts have been made in the past to solve the non-homogeneous form of the equation. Because of the limitations of the analytical solution for certain values of the parameter ν due to slow convergence, this paper discusses a numerical approach which uses the method of lines to solve the homogeneous case and involves finite Fourier series. The method is attractive as it works efficiently and provides good results for $\nu = 1.0$ and results with non-increasing 'energy' for $\nu = 0.1$.

1. Introduction

Historically the homogeneous form of Burgers' equation, namely

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}, \tag{1}$$

where $u = u(x, t)$ in some domain and ν is a parameter, first appeared in a paper by Bateman (1915) when he mentioned it as worthy of study and gave a special solution. Since then this equation has found applications in fields as diverse as number theory, gas dynamics, heat conduction, elasticity, etc. The complete and explicit solution of the equation became known in 1950 (see Hopf 1950). Burgers' equation governs many phenomena (approximately) and is therefore of interest.

In an attempt to analyse the non-homogeneous case Rodin (1970) relates Burgers' partial differential equation to a Riccati ordinary differential equation through a similarity transformation. Via this route, he shows that solutions to the non-homogeneous equation can be obtained. Since then, the numerical method of lines has been developed for the solution of nonlinear partial differential equations.

This approach has been used by Sincovec and Madsen (1975) and tested on Burgers' equation for particular initial and boundary conditions. Roughly speaking, the spatial variable x in the time-dependent PDE is discretised, which results in a semi-discrete approximating system of ODEs. Then ODE methods are used to solve the resulting equations to obtain numerical approximations to the original PDE.

More recently, a simplified Galerkin method has been developed for hyperbolic equations by Chin *et al* (1979). They modify a Galerkin method for nonlinear hyperbolic equations so that it becomes a simpler method of lines which may be viewed as a collocation method. This method has been tested with some success on Burgers' equation for particular initial and boundary conditions.

0305-4470/81/051029+09$01.50

We feel that there is much merit in the use of the method of lines, and have considered its application involving finite Fourier series. In the method discussed we solve a set of ODEs for the amplitudes of the sine terms. This provides a neat alternative to the method used by Sincovec and Madsen (1975) who solve the set of ODEs directly from the discretised equations, using a Runge–Kutta technique.

The method is tested by solving Burgers' equation in an open rectangle where the boundary conditions are

$$u(0, t) = u(1, t) = 0, \qquad t > 0,$$

and the initial condition is

$$u(x, 0) = f(x), \qquad 0 < x < 1.$$

The solution is required in $0 < x < 1$ for $t > 0$ and for illustration $f(x)$ is taken to be $\sin \pi x$. However, the method applies equally well to other choices of initial and boundary conditions.

Good agreement is found for large values of ν by comparing the numerical results with the analytical results. The method works well for values of ν down to 0.1 where the analytical formula is not particularly helpful because of slow convergence. This method of lines also avoids the stability problems associated with the traditional finite-difference techniques. Relevant details of the analytical solution of Burgers' equation are contained in the Appendix.

2. Fourier series approach

First of all we consider the solution of Burgers' equation analytically, using the method of lines. We use the notation $u(x, t_r) = u(x, rk) = u_r(x)$ and approximate $\partial u/\partial t$ by $(u_{r+1}(x) - u_r(x))/k$. This means that equation (1) can be approximated by

$$\frac{u_{r+1} - u_r}{k} = \nu \frac{\mathrm{d}^2 u_{r+1}}{\mathrm{d}x^2} - u_{r+1} \frac{\mathrm{d}u_{r+1}}{\mathrm{d}x}.$$

Hence

$$\frac{\mathrm{d}^2 u_{r+1}}{\mathrm{d}x^2} - \frac{u_{r+1}}{\nu k} = \frac{-u_r}{\nu k} + \frac{u_{r+1}}{\nu} \frac{\mathrm{d}u_{r+1}}{\mathrm{d}x} \qquad (r = 0, 1, 2, \ldots). \tag{2}$$

Taking $r = 0$ and letting $B = 1/\nu k$ gives

$$\frac{\mathrm{d}^2 u_1}{\mathrm{d}x^2} - B u_1 = -B u_0 + \frac{u_1}{\nu} \frac{\mathrm{d}u_1}{\mathrm{d}x}. \tag{3}$$

To first order we neglect the nonlinear term $(u_1/\nu)(\mathrm{d}u_1/\mathrm{d}x)$ and use the initial condition $u_0 = \sin \pi x$. The general solution is

$$u_1 = \alpha \, \mathrm{e}^{\sqrt{Bx}} + \beta \, \mathrm{e}^{-\sqrt{Bx}} + [B/(B + \pi^2)] \sin \pi x, \tag{4}$$

where α and β are constants to be determined.

On applying the boundary conditions

$$u(0, t) = u(1, t) = 0, \qquad t > 0,$$

we obtain the first-order approximation

$$u_1 = [B/(B + \pi^2)] \sin \pi x. \tag{5}$$

Now we iterate by explicitly working out the correction term $(u_1/\nu)(\mathrm{d}u_1/\mathrm{d}x)$ using the linear u_1 and the equation resolved. In this way the correction term can be approximated by $(\pi/2\nu)\sin 2\pi x$. This introduces another term into the particular integral, namely $-[\pi/2(4\pi^2+B)]\sin 2\pi x$. Therefore to second order the solution is

$$u_1 = \frac{B}{B+\pi^2}\sin \pi x - \frac{\pi}{2(4\pi^2+B)}\sin 2\pi x. \tag{6}$$

This process can be repeated to give the approximate solution at higher time steps.

A more systematic approach would be to use the method of lines involving finite Fourier series. The nonlinear terms will generate a solution different from the initial condition input.

Setting $1/\nu = 4A$ in equation (2) we obtain

$$\frac{\mathrm{d}^2 u_{r+1}}{\mathrm{d}x^2} - 4Au_{r+1}\frac{\mathrm{d}u_{r+1}}{\mathrm{d}x} - Bu_{r+1} = -Bu_r \qquad (r = 0, 1, 2, \ldots). \tag{7}$$

Imposing the boundary conditions $u_r(0) = u_r(1) = 0$, and the initial condition $u_0(x) = f(x)$ or some approximation to $f(x)$, leads to a set of nonlinear equations because of the nonlinear term. To the same order of accuracy, equation (7) may be replaced by the linear system

$$\frac{\mathrm{d}^2 u_{r+1}}{\mathrm{d}x^2} - 2Au_{r+1}\frac{\mathrm{d}u_r}{\mathrm{d}x} - 2Au_r\frac{\mathrm{d}u_{r+1}}{\mathrm{d}x} - Bu_{r+1} = -Bu_r \qquad (r = 0, 1, 2, \ldots). \tag{8}$$

As in the case of equation (3), the solution of equation (8) will require only the determination of the particular integral, because of the boundary conditions.

Assuming the particular integral can be expressed in the form

$$\begin{aligned} u_r(x) &= \sum_{j=1}^{M} \alpha_j \sin j\pi x \qquad (\alpha \text{ known}) \\ u_{r+1}(x) &= \sum_{k=1}^{M} \beta_k \sin k\pi x \qquad (\beta \text{ unknown}) \end{aligned} \tag{9}$$

we then substitute into equation (8) to give

$$\begin{aligned} &\sum_{k=1}^{M} (B + k^2\pi^2)\beta_k \sin k\pi x + A\pi \sum_{j=1}^{M}\sum_{k=1}^{M} j\alpha_j\beta_k[\sin(k+j)\pi x + \sin(k-j)\pi x] \\ &\quad + A\pi \sum_{j=1}^{M}\sum_{k=1}^{M} k\alpha_j\beta_k[\sin(j+k)\pi x + \sin(j-k)\pi x] \\ &\quad = B\sum_{j=1}^{M} \alpha_j \sin j\pi x. \end{aligned} \tag{10}$$

Equating coefficients of $\sin i\pi x$ in equation (10) gives

$$\begin{aligned} &(B + i^2\pi^2)\beta_i + A\pi i\left(\sum_{j=1}^{i-1} \alpha_{i-j}\beta_j - \sum_{j=i+1}^{M} \alpha_{j-i}\beta_j - \sum_{j=1}^{M} \alpha_{i+j}\beta_j\right) \\ &\quad = B\alpha_i \qquad (i = 1, 2, 3, \ldots, M). \end{aligned} \tag{11}$$

This means that we have a system of M simultaneous linear equations of the form

$$C\boldsymbol{\beta} = \boldsymbol{\alpha} \tag{12}$$

where the (i, j) coefficient of the matrix C is given by

$$C_{ij} = (1/B)[(B + i^2\pi^2)\delta_{ij} + A\pi i(\alpha_{i-j} - \alpha_{j-i} - \alpha_{i+j})]$$

and

$$\alpha_k = 0 \text{ if } k \leqslant 0,$$

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases}$$

Hence we may compute the coefficients β_i $(i = 1, 2, 3, \ldots, M)$ for any order M.

3. Discussion of results

As a check on the accuracy at the first time step where $t = 0.01$, equations (5) and (6) are evaluated below for the case $\nu = 1$. Taking $k = \delta t = 0.01$, equations (5) and (6) give

$$u(x, 0.01) = [100/(100 + \pi^2)] \sin \pi x \tag{13}$$

and

$$u(x, 0.01) = \frac{100}{100 + \pi^2} \sin \pi x - \frac{\pi}{8(\pi^2 + 25)} \sin 2\pi x \tag{14}$$

respectively.

We first compare results from equations (13) and (14) with those obtained from both an explicit and implicit finite-difference scheme.

An explicit scheme for Burgers' equation is

$$u_{i,j+1} = \nu s(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}) + [1 - \tfrac{1}{2}sh(u_{i+1,j} - u_{i-1,j})]u_{i,j}$$

where $u_{i,j} = u(ih, jk)$ and $s = k/h^2$. Taking $\nu = 1$, $h = 0.25$, $k = 0.01$, which means that $k \leqslant h^2/2$ is satisfied, leads to

$$u_{i,j+1} = 0.16(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}) + [1 - 0.02(u_{i+1,j} - u_{i-1,j})]u_{i,j}.$$

An implicit scheme for Burgers' equation is

$$u_{i,j+1} = u_{i,j} + \nu s(u_{i+1,j+1} - 2u_{i,j+1} + u_{i-j,j+1}) + \tfrac{1}{2}sh(u_{i+1,j+1} - u_{i-1,j+1})u_{i,j+1}.$$

Here the unknown $u_{i,j+1}$ is involved in a nonlinear expression and therefore we must use an iterative solution. Taking $\nu = 1$, $h = 0.25$, $k = 0.01$ as before leads to

$$u_{i,j+1} = u_{i,j} + 0.16(u_{i+1,j+1} - 2u_{i,j+1} + u_{i-j,j+1}) - 0.02(u_{i+1,j+1} - u_{i-1,j+1})u_{i,j+1}.$$

A comparison of the results at the first time step $t = 0.01$ is then presented in table 1.

Table 1. Comparison of results at $t = 0.01$ where $h = 0.25$, $k = 0.01$, $\nu = 1$.

Method	$u(0.25)$	$u(0.50)$	$u(0.75)$
Lines (1st order)	0.6436	0.9102	0.6436
Lines (2nd order)	0.6323	0.9102	0.6548
Explicit	0.6266	0.9062	0.6549
Implicit	0.6377	0.9142	0.6556
Analytic	0.6290	0.9057	0.6524

Clearly there is reasonable agreement and this suggests that it would certainly be worthwhile considering the Fourier series approach with a larger number of terms.

A computer program which uses a matrix inversion package to invert the matrix C has been written for this scheme, and the results obtained for $t = 0\ (0.01)\ 0.20$ for the case $\nu = 1$ and $M = 4$ are presented in table 2. The energy criterion of multiplying Burgers' equation throughout by u and integrating which was suggested by Cole (1951) has been included in the computer program and used to monitor the results. These results are closer to the analytical results than those obtained by the explicit or implicit finite-difference methods discussed earlier, and agree with the analytical results to two decimal places even in the $M = 4$ case over the range $t = 0$ to 0.20. Further improvement can be obtained either by using larger M or by using smaller k.

Table 2. Solutions for the case $M = 4$ with $\nu = 1$, $k = 0.01$.

t	$u(0.25)$	$u(0.50)$	$u(0.75)$	Energy
0.01	0.6334	0.9100	0.6539	4.0895
0.02	0.5700	0.8278	0.6017	3.3896
0.03	0.5149	0.7531	0.5516	2.8091
0.04	0.4663	0.6851	0.5043	2.3274
0.05	0.4232	0.6233	0.4600	1.9278
0.06	0.3847	0.5671	0.4191	1.5964
0.07	0.3501	0.5161	0.3813	1.3218
0.08	0.3188	0.4696	0.3467	1.0944
0.09	0.2905	0.4274	0.3150	0.9060
0.10	0.2648	0.3890	0.2861	0.7501
0.11	0.2415	0.3540	0.2599	0.6210
0.12	0.2202	0.3222	0.2360	0.5142
0.13	0.2009	0.2932	0.2143	0.4257
0.14	0.1832	0.2669	0.1946	0.3525
0.15	0.1671	0.2429	0.1767	0.2919
0.16	0.1524	0.2211	0.1605	0.2417
0.17	0.1390	0.2012	0.1458	0.2002
0.18	0.1267	0.1832	0.1324	0.1658
0.19	0.1155	0.1667	0.1203	0.1373
0.20	0.1053	0.1517	0.1093	0.1137

The size of the diagonal term in the coefficient matrix C of equation (12) is given by $(1/B)(B + i^2\pi^2 - A\pi i\alpha_{2i})$. As B is inversely proportional to the time step and A is constant, a value of the time step can always be chosen which ensures diagonal dominance in the coefficient matrix C and this provides another possible method of solution.

Up to this stage we have considered the case $\nu = 1$. However, smaller values of ν present much more of a challenge in the solution of Burgers' equation. For this reason results are presented in table 3 for the case $\nu = 0.1$, $M = 4$.

We have already mentioned that the energy criterion has been used to monitor the results, and as can be seen from the results the energy decreases at each step as expected, provided $\nu \geqslant 0.1$. A decrease in energy as t increases implies that $u(x, t)$ is correct. For $\nu = 0.01$ a similar study shows that this method is not satisfactory, as the energy appears to increase at each step. For large values of ν good results can be obtained by using only a few Fourier terms. However, as ν becomes smaller the β

Table 3. Solution for the case $M = 4$ with $\nu = 0.1$, $k = 0.01$.

t	$u(0.25)$	$u(0.50)$	$u(0.75)$	Energy
0.01	0.6854	0.9898	0.7153	0.4842
0.02	0.6649	0.9789	0.7229	0.4757
0.03	0.6455	0.9673	0.7300	0.4678
0.04	0.6272	0.9553	0.7364	0.4605
0.05	0.6099	0.9429	0.7421	0.4537
0.06	0.5935	0.9303	0.7472	0.4473
0.07	0.5780	0.9173	0.7517	0.4412
0.08	0.5632	0.9043	0.7554	0.4353
0.09	0.5493	0.8912	0.7585	0.4297
0.10	0.5360	0.8780	0.7609	0.4243
0.11	0.5234	0.8648	0.7626	0.4189
0.12	0.5114	0.8518	0.7637	0.4137
0.13	0.4999	0.8388	0.7642	0.4085
0.14	0.4890	0.8259	0.7641	0.4034
0.15	0.4785	0.8132	0.7634	0.3982
0.16	0.4685	0.8007	0.7621	0.3930
0.17	0.4589	0.7884	0.7603	0.3878
0.18	0.4498	0.7762	0.7581	0.3825
0.19	0.4410	0.7643	0.7553	0.3772
0.20	0.4325	0.7527	0.7522	0.3718

coefficients increase in magnitude and therefore a large number of Fourier terms are required to obtain the same accuracy. This is clearly demonstrated by examining the results in tables 4 and 5. The disadvantage is that a large number of Fourier terms will lead to inversion of large matrices. The method of lines certainly enables us to avoid stability problems, but this has to be balanced against the introduction of a large (ν-dependent) matrix which has to be inverted.

However, by drawing up tables of β_k against time it can be seen from the results at each time step that the Fourier coefficients only change by a small amount. This means that a process of updating the inverse matrix is possible and would avoid determining the inverse of a large matrix from the same starting point at each stage. Therefore we feel that this Fourier method is attractive as it works efficiently and provides good results.

Table 4. Computed values of the β coefficients for various values of M for the case $\nu = 0.1$, $t = 0.20$ using $k = 0.01$.

M	2	4	6	8
β_1	0.7953	0.7952	0.7952	0.7952
β_2	−0.1685	−0.1598	−0.1598	−0.1598
β_3		0.0425	0.0425	0.0425
β_4		−0.0129	−0.0123	−0.0123
β_5			0.0036	0.0036
β_6			−0.0011	−0.0011
β_7				0.0003
β_8				−0.0001

Table 5. Computed values of the β coefficients for various values of M for the case $\nu = 0.01$, $t = 0.20$ using $k = 0.01$.

M	2	4	6	8
β_1	0.9358	0.9350	0.9349	0.9349
β_2	−0.2864	−0.2607	−0.2605	−0.2605
β_3		0.1067	0.1065	0.1065
β_4		−0.0587	−0.0509	−0.0508
β_5			0.0264	0.0263
β_6			−0.0169	−0.0143
β_7				0.0081
β_8				−0.0055

In our method the coefficient β_k rapidly became smaller as k increases for large values of ν ($\nu = 1$ say), but not so if ν is small. This indicates the development of a wave front which restricts the value of ν because the number of necessary Fourier terms becomes large. As already mentioned, the method is unsuitable for $\nu \leqslant 0.01$, and the solutions indicate that a piecewise polynomial approximation (i.e. finite element) should be attempted where the size of the elements should be chosen to take into account the nature of the solution.

Acknowledgment

The authors would like to thank Mr A E Cook (Teesside Polytechnic) for both his help and interest in this work.

Appendix: analytical solution of Burgers' equation

The exact solution of Burgers' equation under the boundary conditions

$$u(0, t) = u(1, t) = 0, \qquad t > 0,$$

and initial condition

$$u(x, 0) = f(x), \qquad 0 \leqslant x \leqslant 1,$$

is given by (see Cole 1951)

$$u(x, t) = \frac{2\pi\nu \sum_{m=1}^{\infty} mA_m \sin m\pi x \exp(-m^2\nu\pi^2 t)}{A_0 + \sum_{m=1}^{\infty} A_m \cos m\pi x \exp(-m^2\nu\pi^2 t)} \tag{A1}$$

where

$$A_m = 2\int_0^1 \cos m\pi x \exp\left(-\frac{1}{2\nu}\int_0^x f(x')\,dx'\right) dx \qquad (m = 1, 2, 3, \ldots),$$

$$A_0 = \int_0^1 \exp\left(-\frac{1}{2\nu}\int_0^x f(x')\,dx'\right) dx.$$

For our case $f(x) = \sin \pi x$,

$$A_m = 2 \exp(-1/2\pi\nu) I_m(1/2\pi\nu) \qquad (m = 1, 2, 3, \ldots),$$

$$A_0 = \exp(-1/2\pi\nu) I_0(1/2\pi\nu),$$

where the modified Bessel functions are

$$I_m\left(\frac{1}{2\pi\nu}\right) = \int_0^1 \cos m\pi x \exp\left(\frac{\cos \pi x}{2\pi\nu}\right) \mathrm{d}x \qquad (m = 1, 2, 3, \ldots),$$

$$I_0\left(\frac{1}{2\pi\nu}\right) = \int_0^1 \exp\left(\frac{\cos \pi x}{2\pi\nu}\right) \mathrm{d}x.$$

Hence

$$u(x, t) = \frac{4\pi\nu \sum_{m=1}^{\infty} \exp(-m^2\nu\pi^2 t) m I_m(1/2\pi\nu) \sin m\pi x}{I_0(1/2\pi\nu) + 2\sum_{m=1}^{\infty} \exp(-m^2\nu\pi^2 t) I_m(1/2\pi\nu) \cos m\pi x}. \qquad \text{(A2)}$$

Provided ν is sufficiently large a reasonable approximation is obtained by taking $I_0 = 1$, $I_1 = 1/4\pi\nu$, $I_2 = I_3 = \ldots = 0$. This leads to the useful approximation

$$u(x, t) \simeq \frac{\sin \pi x \exp(-\nu\pi^2 t)}{1 + (1/2\pi\nu) \cos \pi x \exp(-\nu\pi^2 t)} \qquad (t > 0). \qquad \text{(A3)}$$

Note the asymmetry about $x = \frac{1}{2}$ which disappears for large values of t.

From the definitions of I_0 and I_m much more accurate expressions than equation (A3) can be found. For the case $\nu = 1$ the leading coefficients to eight decimal places are as follows:

$$\begin{aligned}
I_0 &= 1.006\,342\,61,\\
I_1 &= 0.079\,829\,68,\\
I_2 &= 0.003\,172\,98,\\
I_3 &= 0.000\,084\,12,\\
I_4 &= 0.000\,001\,67,\\
I_5 &= 0.000\,000\,03,\\
I_6 = I_7 = \ldots &= 0.000\,000\,00.
\end{aligned}$$

However, for values of ν below 1 the convergence of equation (A1) is slow, as the coefficients I_m drop off much less rapidly than for the $\nu = 1$ case. This certainly applies for values of ν in the region of 0.1. For this reason it is important to consider possible efficient and accurate numerical techniques which produce solutions to Burgers' equation for such values of ν. Such a technique, which works well for values of ν down to 0.1, is discussed in § 2.

In the limit as $\nu \to 0$ equation (A1) approaches

$$u(x, t) = \frac{4\pi\nu \sum_{m=1}^{\infty} m \sin m\pi x \exp(-m^2\nu\pi^2 t)}{1 + 2\sum_{m=1}^{\infty} \cos m\pi x \exp(-m^2\nu\pi^2 t)}.$$

This is not so worrying, as this result can be simply expressed in terms of the θ-functions mentioned by Cole (1951). Numerous transformations for these exist, and these can be simply used to achieve better 'convergence'. Hence the difficulties arise not for values of ν close to zero but for larger values, particularly in the range 0.01 to 1.

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